# MATH 3060 Tutorial 11

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1. Let $\{f_n : \mathbb{C} \to \mathbb{C}\}$ be a sequence of functions, and let $U$ be a bounded open subset of $\mathbb{C}$.
   (a) Assume $U$ is a disc and suppose $\{f_n\}$ and partial derivatives of $f$ are uniformly bounded on $U$, show that $\{f_n\}$ is precompact.
   (b) Suppose $\{f_n\}$ is uniformly bounded on $U$ and each $f_n$ is holomorphic, show that there exists a subsequence $f_{n_k}$ with converges pointwise to a continuous function, show that the convergence is uniform on every compact subset of $U$.
2. Prove that the dimension of a Banach space is either finite or uncountable.
3. Let $X = (C[0, 2\pi], ||\cdot||_\infty)$, it is a Banach space. Let $F_n : X \to \mathbb{C}$ be

$$F_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\frac{\sin(n+\frac{1}{2})x}{\sin\frac{1}{2}x}dx.$$

   (a) For each $n \in \mathbb{Z}_+$, show that there exists an $f \in X$ with

$$\frac{1}{2\pi}\int |f| = 1,\ F_n(f) > \frac{2}{\pi^2}\sum_{k<n}\frac{1}{k}.$$

   (b) Using Baire category theorem, show that there exists an $f \in X$ whose Fourier series diverges at the point 0.
4. Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function such that for each $x \in \mathbb{R}$, there exists some $n \in \mathbb{Z}_+$ with $f^{(n)}(x) = 0$. Show that $f$ is a polynomial.
   See here for a solution.